# A Propositional Logic That Handles Conditional Probabilities

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**Abstract**

In [1], two logics for reasoning about probabilities were introduced. The first was unable to reason about conditional probabilities. To rectify this [1] introduce their second probabilistic logic. However, unlike their first, they find it necessary to introduce quantifiers. In this paper we rectify this *defect* by presenting a *propositional* logic that can reason about conditional probabilities. Our method of proof is such that we may generalize it to encapsulate soundness and completeness for numerous other logics (eg. the first probabilistic logic of [1]; fuzzy logic etc.). Our main requirement for these generalizations is that our logics should have a first-order metatheory.

## 1 Introduction

The primary purpose of this paper to present a sound and complete axiomatization for a probabilistic propositional logic, that is capable of arguing about conditional probabilities. Such a logic was first presented in [1], however the logic presented there is quite different from that presented here in two main ways.

Firstly, their soundness and completeness result was obtained at the cost of allowing in quantifiers. Here we expand our language slightly by allowing in reciprocation ($^{-1}$), and thus obtain a soundness and completeness result for a propositional probabilistic logic that handles conditional probabilities (note that this result is contrary to a remark in section 6 of [1]). Why such a (small?) change in our language should achieve this result is related to our next difference.

The techniques used in [1] make use of elimination of quantifiers for the first-order theory of real-closed fields. Here we extract from this theory the precise model-theoretic properties we require and combine these properties with a *good axiomatization* for our probabilistic logic. What we shall mean by a *good axiomatization* shall be made clearer at a latter point in this paper. For the moment it suffices to say that an axiomatization will be considered *good* if it has a *well-behaved* proof theory (ie. cut-elimination, subformulae property etc.).

One final (and minor) difference is that we shall work with a Gentzen-style axiomatization rather than a Hilbert-style axiomatization throughout. This style is chosen since its proof-theoretic properties are much easier to work with than in a Hilbert-style axiomatization. The reader is referred to [4] for more about this style of axiomatization.

Section 2 describes the bulk of the work needed to prove our soundness and completeness result. It is in this section that we shall make the main use of our model and proof-theoretic properties in proving this result. Section 3 proves our soundness and completeness result proper and builds heavily on the work of section 2. The main techniques used in this section consist in simply showing, that our probabilistic logic may be interpreted in a suitably *nice* fashion, in the setting of section 2.

Section 4 concludes the paper by indicating how the techniques of this paper may be generalized to prove soundness and completeness results for many other logics (eg. fuzzy logic, the probabilistic logic of [1], etc.). We also comment on the limitations of our method and how known model-theoretic (eg. definability, complexity, etc.) and proof-theoretic properties (eg. resolution, unification, etc.) may be transferred into these *new* logics.

## 2 The Basic Soundness and Completeness Result

The main aim of this section is to show the following theorem;

**Theorem 1.** *For each quantifier-free $\theta$ (formulated in language $\mathcal{L}_{\mathsf{Q}}$ );*

*$\models |\theta$ iff $|\theta$ is provable from $\sigma(OF)$ using just structural and propositional rules of inference*

Unfamiliar terms and notations are explained in the following definition.

**Definition 1.** *Let $\mathcal{L}_{\mathsf{Q}} = \{+, -, ., ^{-1}, \leq, \doteq\} \cup \{\underline{q} | q \in \ \}$ be the language of ordered fields with equality and all rationals as constants, and define the following;*

- *and are the standard first-order structures (over the language $\mathcal{L}_{\mathsf{Q}}$) of the real and rational ordered fields.*
- *$OF$ is the theory of ordered fields (formulated in the language $\mathcal{L} = \{+, -, ., ^{-1}, \leq, \doteq, \underline{0}, \underline{1}\}$) and consists of the following sentences;*
  $\forall x, y, z(x + (y + z) \doteq (x + y) + z)$
  $\forall x, y(x + y \doteq y + x)$
  $\forall x(x + \underline{0} \doteq x)$

$\forall x(x + (-x) \doteq \underline{0})$
$\forall x, y, z(x.(y.z) \doteq (x.y).z)$
$\forall x, y(x.y \doteq y.x)$
$\forall x(x.\underline{1} \doteq x)$
$\forall x(x \doteq \underline{0} \vee x.x^{-1} \doteq \underline{1})$
$\forall x, y, z(x.(y + z) \doteq (x.y) + (x.z))$
$\underline{0} \neq \underline{1}$
$\underline{0} \leq \underline{1}$
$\forall x, y, z(x \leq y \Longrightarrow x + z \leq y + z)$
$\forall x, y(\underline{0} \leq x \wedge \underline{0} \leq y \Longrightarrow \underline{0} \leq x.y)$
$\forall x(x \leq x)$
$\forall x, y(x \leq y \wedge y \leq x \Longrightarrow x \doteq y)$
$\forall x, y(x \leq y \vee y \leq x)$
$\forall x, y, z(x \leq y \wedge y \leq z \Longrightarrow x \leq z)$
$\forall x(x \doteq x)$
$\forall x, y, z, w(x \doteq y \wedge z \doteq w \wedge x \doteq z \Longrightarrow y \doteq w)$
$\forall x, y, z, w(x \doteq y \wedge z \doteq w \wedge x \leq z \Longrightarrow y \leq w)$
$\forall x, y, z, w(x \doteq y \wedge z \doteq w \Longrightarrow x + z \doteq y + w)$
$\forall x, y, z, w(x \doteq y \wedge z \doteq w \Longrightarrow x.z \doteq y.w)$
$\forall x, y(x \doteq y \Longrightarrow -x \doteq -y)$
$\forall x, y(x \doteq y \Longrightarrow x^{-1} \doteq y^{-1})$
*Where we assume here and elsewhere in this paper that bracketing for expressions such as $\theta_1 \vee \ldots \vee \theta_n$ or $\theta_1 \wedge \ldots \wedge \theta_n$ associate to the left.*

- $\sigma(OF) = \{\theta(\overrightarrow{t}) | \overrightarrow{t}$ *is a tuple of terms formulated in* $\mathcal{L}_{\mathsf{Q}}$ , $\forall \overrightarrow{x} \theta(x) \in OF$ *and* $\theta(\overrightarrow{x})$ *is quantifier-free* $\}$ *is the set of all members of* $OF$, *with leading universal quantifiers dropped and the resulting free variables substituted by arbitrary terms of the language* $\mathcal{L}_{\mathsf{Q}}$.

- $\vdash$ *is the provability predicate for the Gentzen-style axiomatization of the first-order predicate calculus (formulated over some fixed language) as presented in (say) [4].*

- *We say that,*

$\Gamma|\Delta$ *is provable from* $T \subseteq P\mathcal{L}$ ($T \vdash \Gamma|\Delta$)
*iff* $\Gamma|\Delta$ *is provable using the axioms and rules of* $\vdash$, *along with the additional axioms* $|\phi$ (*for* $\phi \in T$)

Before proving theorem 1 we shall first need to establish three lemmas.

**Lemma 1.** *For each universal formulae $\theta$ (formulated in $\mathcal{L}_{\mathsf{Q}}$);*

$$\models |\theta \text{ iff } OF \cup \Delta() \models |\theta$$

*Where $\Delta()$ is the diagram of*

*ie. set of all atomic and negated atomic sentences (in the language $\mathcal{L}_{\mathsf{Q}}$) true in*

*Proof.* Here the essential model-theoretic property we require of  is that it is an existentially-closed model over the theory $OF$. It turns out that this property is the only model-theoretic property we shall need to exploit. The reader may either take this lemma as the definition of what it is for  being existentially-closed over $OF$ or may alternately consult [3] for equivalent definitions of this concept. □

**Lemma 2.** *For each quantifier-free formulae $\theta$ (formulated in $\mathcal{L}_{\mathsf{Q}}$);*

$$OF \cup \Delta() \vdash |\theta \text{ iff } \sigma(OF) \cup \Delta() \vdash |\theta$$

*Proof.* Here we exploit our first proof-theoretic property, Herbrand's theorem. Herbrand's theorem states that;

for $\exists x\theta(x)$ an existential formula such that $\vdash |\exists x\theta(x)$, then there are terms $t_1, \ldots, t_n$ such that;
$\vdash |\theta(t_1), \ldots, \theta(t_n)$

The reader is referred to [4] for a proof of this result. We see that this theorem is of some use by noting the following;

$OF$ is a universal theory, $\Delta()$ consists of atomic sentences (which trivially contain no quantifiers) and $\theta$ is quantifier-free.

Our required result now follows from this and Herbrand's theorem since;

$OF \cup \Delta() \vdash |\theta$
iff $\exists\phi_1, \ldots, \phi_n \in OF \cup \Delta() \cdot \vdash \phi_1, \ldots, \phi_n|\theta$ ,by definition
iff $\exists\phi_1, \ldots, \phi_n \in OF \cup \Delta() \cdot \vdash |\phi_1 \wedge \ldots \wedge \phi_n \implies \theta$ ,by propositional reasoning

iff $\exists \phi_1, \ldots, \phi_n \in \sigma(OF) \cup \Delta() \cdot \vdash |\forall \overrightarrow{x}(\phi_1 \wedge \ldots \phi_n) \Longrightarrow \theta$ ,by above comment and quantificational reasoning

Note that in the following, by suitable changes of bound variables, we can arrange for $\overrightarrow{x}$ not to occur free anywhere in $\theta$.

iff $\exists \phi_1, \ldots, \phi_n \in \sigma(OF) \cup \Delta() \cdot \vdash |\exists \overrightarrow{x}(\phi_1 \wedge \ldots \wedge \phi_n \Longrightarrow \theta)$ ,by above comment and quantificational reasoning

iff $\exists \phi_1, \ldots, \phi_n \in \sigma(OF) \cup \Delta() \cdot \vdash |\phi_1 \wedge \ldots \wedge \phi_n \Longrightarrow \theta$ ,by several uses of Herbrand's theorem, propositional reasoning and $\exists - R$

iff $\exists \phi_1, \ldots, \phi_n \in \sigma(OF) \cup \Delta() \cdot \vdash \phi_1, \ldots, \phi_n | \theta$ ,by propositional reasoning

iff $\sigma(OF) \cup \Delta() \vdash |\theta$ □

**Lemma 3.** *For each quantifier-free formulae $\theta$ (formulated in $\mathcal{L}_{\mathsf{Q}}$);*

*$\sigma(OF) \cup \Delta() \vdash |\theta$ iff there is a proof of $|\theta$ from $\sigma(OF)\Delta()$ that only uses structural and propositional rules*

*Proof.* Here we exploit the subformula property for our axiomatization of the predicate calculus (again see [4]). We now simply note that by the definition of $\sigma(OF)$, $\Delta()$ and the choice of $\theta$, we can have no occurrences of quantifiers and so no quantifier rules being used in our proof. □

*Proof of Theorem 1.* Let $\theta$ be a quantifier-free formulae formulated in $\mathcal{L}_{\mathsf{Q}}$ . Now,

$$\models |\theta \text{ implies } \models |\theta \text{ ,since } \subseteq \text{ and } \theta \text{ is quantifier-free}$$

But we also have that $\models OF \cup \Delta()$, and so that by lemma 3 that if $\models |\theta$ holds then;

$$OF \cup \Delta() \models |\theta \text{ and so } \models |\theta$$

Hence,

$\models |\theta$

iff $\models |\theta$

iff $OF \cup \Delta() \models |\theta$ ,by lemma 3

iff $OF \cup \Delta() \models |\theta$ ,by soundness and completeness of $\vdash$ (see [4])

iff $\sigma(OF) \cup \Delta() \vdash |\theta$ ,by lemma 4

iff there is a proof of $|\theta$ from $\sigma(OF) \cup \Delta()$ that uses structural and propositional rules ,by lemma 5

Notice now that by propositional reasoning we have the following obvious corollary to theorem 1;

**Corollary 1.** *Let* $\Gamma$, $\Delta$ *be finite lists of quantifier-free formulae formulated in* $\mathcal{L}_{\mathsf{Q}}$ *. Then;*

$$\models \Gamma|\Delta \text{ iff } \sigma(OF) \cup \Delta() \vdash \Gamma|\Delta$$

## 3 Soundness and Completeness for a Conditional Probabilistic Logic

The aim of this section shall be to prove a soundness and completeness result for a probabilistic logic that is capable of reasoning with conditional probabilities. As already mentioned our result differs from that in [1] in two main ways. Firstly, by allowing reciprocation ($^{-1}$) into our language we may eliminate any need for quantifiers in our axiomatization. Secondly, our method of proof is quite different from that in [1]. Using our method we are able to encapsulate not just one soundness and completeness result but many! This is something we discuss further in section 4.

We proceed by formalizing our meta-theory and then exploiting known model and proof-theoretic facts to obtain soundness and completeness. Section 2 has already established the properties of our meta-theory that we shall be interested in. All that remains for us to do in this section is to show how these properties may be exploited. First we define our probability logic and give its axiomatization, which we shall then go on to show is sound and complete.

**Definition 2.** *Let* $\ = \{p_i | i \in \ \}$, $\vdash_{SC}$ *be the provability predicate for the propositional calculus and define the following;*

- $S$ *is the set of boolean sentences generated by the following grammar;*

$$\psi ::= p_i | \theta \underline{\wedge} \phi | \theta \underline{\vee} \phi | \theta \underline{\Longrightarrow} \phi | \theta \underline{\Longleftrightarrow} \phi | \underline{\neg} \theta$$

  *where* $p \in$

- $T$ *is the set of terms generated by the following grammar;*

$$t ::= \underline{q} | m(a) | r + s | - r | r.s | r^{-1}$$

*where* $q \in$ , $a \in S$

- $F$ *is the set of formulae generated by the following grammar;*

$$\psi ::= r \leq s | r \doteq s | \theta \wedge \phi | \theta \vee \phi | \theta \Longrightarrow \phi | \neg\theta$$

*where* $r, s \in T$

- $w : S \longrightarrow [0,1]$ *is a probability function iff the following hold,*
  - *for* $\theta \in S$, $\vdash_{SC} \theta$ *implies* $w(\theta) = 1$ *and* $w(\underline{\neg}\theta) = 0$
  - *for* $\theta, \phi \in S$, $\vdash_{SC} \theta \underline{\Longleftrightarrow} \phi$ *implies* $w(\theta) = w(\phi)$
  - *for* $\theta, \phi \in S$, $w(\theta) + w(\phi) = w(\theta \underline{\vee} \phi) + w(\theta \underline{\wedge} \phi)$

- $\imath_w : T \longrightarrow$ *(for* $w$ *a probability function) is given by,*
  $\imath_w(m(a)) = w(a)$
  $\imath_w(\underline{q}) = q$
  $\imath_w(r + s) = \imath_w(r) + \imath_w(s)$
  $\imath_w(-r) = -\imath_w(r)$
  $\imath_w(r.s) = \imath_w(r).\imath_w(s)$
  $\imath_w(r^{-1}) = \imath_w(r)^{-1}$
  *where* $a \in S$, $q \in$ *and* $r, s \in T$

- $\models_{CPL}$ *is determined by the following;*
  $w \models_{CPL} \theta \wedge \phi$ *iff* $w \models_{CPL} \theta$ *and* $w \models_{CPL} \phi$
  $w \models_{CPL} \theta \vee \phi$ *iff* $w \models_{CPL} \theta$ *or* $w \models_{CPL} \phi$
  $w \models_{CPL} r \leq s$ *iff* $\imath_w(r) \leq \imath_w(s)$
  $w \models_{CPL} r \doteq s$ *iff* $\imath_w(r) = \imath_w(s)$
  *where* $r, s \in T$, $\theta, \phi \in F$ *and* $w$ *is a probability function.*
  *and* $w \models_{CPL}$ *is extended to Gentzen-style sequents as follows;*

$$w \models_{CPL} \Gamma|\Delta \text{ iff } (\forall\theta \in \Gamma \cdot w \models_{CPL} \theta) \text{ implies } (\exists\phi \in \Delta \cdot w \models_{CPL} \phi$$

  *We also say that,*

$$\models_{CPL} \Gamma|\Delta \text{ iff } \forall \text{ probability function } w : S \longrightarrow [0,1] \cdot w \models_{CPL} \Gamma|\Delta$$

- *if* $a \in S$ *then,*

$$a \text{ is an } n\text{-atom iff } a \in \{\delta(p_0) \wedge \ldots \wedge \delta(p_{n-1}) | \text{ for } j = 0, \ldots, n-1, \delta(p_j) \in \{p_j, \neg p_j\}\}$$

*- recall that we have assumed* $\wedge$ *is left associative.*

- $At_n = \{a \in S | a$ *is an n-atom* $\}$ *and* $At = \bigcup_{n \in \mathsf{N}} At_n$
- $\theta \in F$ *is n-normal iff* $\forall a \in S \cdot m(a)$ *occurs in* $\theta$ *implies* $a \in At_n$
- *A finite list* $\Gamma$ *of members of* $F$ *is n-normal iff* $\forall \theta$ *occurring in* $\Gamma \cdot \theta$ *is n-normal*
- *A list* $\Gamma$ *of members of* $F$ *is normal iff* $\exists n \in \ \cdot \Gamma$ *is n-normal*
- $\vdash_{CPL}$ *is the provability predicate for the following axiomatization;*

**Axioms**

**Logical**

$r \leq s | r \leq s$
$r \doteq s | r \doteq s$

**Equality**

$| r \doteq r$
$r \doteq s, t \doteq u, r \doteq t | s \doteq u$
$r \doteq s, t \doteq u, r \leq t | s \leq u$
$r \doteq s, t \doteq u | r + t \doteq s + u$
$r \doteq s, t \doteq u | r.t \doteq s.u$
$r \doteq s | -r \doteq -s$
$r \doteq s | r^{-1} \doteq s^{-1}$

**Ordering**

$| r \leq r$
$r \leq s, s \leq r | r \doteq s$
$| r \leq s, s \leq r$
$r \leq s, s \leq t | r \leq t$
$r \leq s | r + t \leq s + t$
$\underline{0} \leq r, \underline{0} \leq s | \underline{0} \leq r.s$

**Field**

$| r + (s + t) \doteq (r + s) + t$
$| r + s \doteq s + r$
$| r + \underline{0} \doteq r$
$| r + (-r) \doteq \underline{0}$

$|r.(s.t) \doteq (r.s).t$
$|r.s \doteq s.r$
$|r.\underline{1} \doteq r$
$|r \doteq \underline{0}, r.r^{-1} \doteq \underline{1}$
$|r.(s+t) \doteq (r.s) + (r.t)$

**Probability**

$|\underline{0} \leq m(a)$
$|m(a) \doteq \underline{1}$ *,if* $\vdash_{SC} a$
$|m(\underline{\neg} a) \doteq \underline{0}$ *,if* $\vdash_{SC} a$
$|m(a) \doteq m(b)$ *,if* $\vdash_{SC} a \underline{\Longleftrightarrow} b$
$|m(a) + m(b) \doteq m(a \underline{\vee} b) + m(a \underline{\triangle} b)$

**Diagram**

$|\underline{p} \doteq \underline{q}$ *,if* $p = q$
$|\underline{p} \leq \underline{q}$ *,if* $p \leq q$
$\underline{p} \doteq \underline{q}|$ *,if* $p \neq q$
$\underline{p} \leq \underline{q}|$ *,if* $p > q$

**Rules**

*Structural and propositional rules of definition 2.*

*Where* $\theta, \phi \in F$, $a, b \in S$, $p, q \in$ , $r, s, t, u \in T$ *and all finite lists in sequents are taken to be members of* $F$

**Theorem 2.** *Let* $\Gamma \cup \Delta$ *be a finite list of members of* $F$*. Then there is a finite list* $\Gamma' \cup \Delta'$ *of members of* $F$ *such that,*

$$\Gamma' \cup \Delta' \text{ is normal}$$
$$\vdash_{CPL} \Gamma|\Delta \text{ iff } \vdash_{CPL} \Gamma'|\Delta'$$

*and,*

$$\models_{CPL} \Gamma|\Delta \text{ iff } \models_{CPL} \Gamma'|\Delta'$$

*Proof.* See [1] □

We thus see that by restricting our attention to normal sequents we loose nothing in terms of provability and satisfiability.

Before we prove our soundness and completeness theorem it is first necessary to show how we may use the results of section 2. The following definition provides the necessary apparatus for this.

**Definition 3.**

- *let* $()^{\#} : T \not\longrightarrow T\mathcal{L}_{\mathsf{Q}}$ *be the partial map given by,*
  $(m(a))^{\#} = x_a$ *,for* $a \in At$
  $(\underline{q})^{\#} = \underline{q}$
  $(r + s)^{\#} = r^{\#} + s^{\#}$
  $(r.s)^{\#} = r^{\#}.s^{\#}$
  $(-r)^{\#} = -r^{\#}$
  $(r^{-1})^{\#} = (r^{\#})^{-1}$
  *where* $r, s \in T$ *and* $q \in$
- *let* $()^{*} : F \not\longrightarrow P\mathcal{L}_{\mathsf{Q}}$ *be the partial map given by,*
  $(r \leq s)^{*} = r^{\#} \leq s^{\#}$
  $(r \doteq s)^{*} = (r^{\#} \doteq s^{\#})$
  $(\theta \wedge \phi)^{*} = \theta^{*} \wedge \phi^{*}$
  $(\theta \vee \phi)^{*} = \theta^{*} \vee \phi^{*}$
  $(\theta \Longrightarrow \phi)^{*} = \theta^{*} \Longrightarrow \phi^{*}$
  $(\neg\theta)^{*} = \neg\theta^{*}$
  *where* $r, s \in T$ *and* $\theta, \phi \in F$
- *For* $w : At_n \longrightarrow [0, 1]$ *such that for* $\theta, \phi \in At_n$*,*
  - *if* $\vdash_{SC} \theta$ *then* $w(\theta) = 1$ *and* $w(\underline{\neg}\theta) = 0$
  - *if* $\vdash_{SC} \theta \underline{\Longleftrightarrow} \phi$ *then* $w(\theta) = w(\phi)$
  - $w(\theta) + w(\phi) = w(\theta \underline{\vee} \phi) + w(\theta \underline{\wedge} \phi)$

  *let* $\overline{w} : S \longrightarrow [0, 1]$ *be the probability function determined by,*

  $$\overline{w}(\theta) = \Sigma_{\phi \in At_n \, and \vdash_{SC} \phi \underline{\Longrightarrow} \theta} w(\phi)$$

  *where empty summands are taken to be zero and* $\theta \in S$*.*

**Theorem 3.** *Let* $\Gamma \cup \Delta$ *be an n-normal, finite list of members of* $F$*. Then,*

$$\vdash_{CPL} \Gamma|\Delta \text{ iff } \models_{CPL} \Gamma|\Delta$$

*Proof.*
$\models_{CPL} \Gamma|\Delta$
iff $\forall$ probability function $w : S \longrightarrow [0, 1] \cdot w \models_{CPL} \Gamma|\Delta$ ,by definition of $\models_{CPL}$
iff $\forall w : At_n \longrightarrow [0, 1] \cdot \overline{w} \models_{CPL} \Gamma|\Delta$ ,since $\Gamma \cup \Delta$ is n-normal
iff $\forall \imath : Var \longrightarrow [0, 1] \cdot \Sigma_{a \in At_n} \imath(x_a) = 1 \Longrightarrow , \imath \models \Gamma^{*}|\Delta^{*}$
Now, taking $\Theta$ to be the sentence,

$$(\wedge_{a \in At_n} \underline{0} \leq m(a)) \wedge \Sigma_{a \in At_n} m(a) \doteq \underline{1}$$

we have that,

iff $\forall \imath : Var \longrightarrow \cdot, \imath \models \Theta^*, \Gamma^* | \Delta^*$
iff $\sigma(OF) \vdash_{CPL} \Theta^*, \Gamma^* | \Delta^*$ ,by corollary 1
iff $\vdash_{CPL} \Theta, \Gamma | \Delta$ ,by fact that $\vdash_{CPL}$ *extends* $\vdash$ (under the partial map $()^*$ ) and members of $\sigma(OF)$ are represented by *sequents* of $\vdash_{CPL}$
iff $\vdash_{CPL} \Gamma | \Delta$ ,since by [1] $\vdash_{CPL} \Theta$ □

**Corollary 2.** *For* $\Gamma, \Delta$ *finite lists of members of* $F$*,*

$$\vdash_{CPL} \Gamma | \Delta \text{ iff } \models \Gamma | \Delta$$

*Proof.* Follows easily from theorems 7 and 9 □

# 4 Conclusions

We have presented a sound and complete axiomatization for a probabilistic logic that is capable of reasoning about conditional probabilities. Of interest to us in this section are the questions of how we may generalize our result, what are its limitations and what more we may learn about our logic by our proof for soundness and completeness. First though, it is of some use to outline how our proof proceeded.

1. Chose a particular meta-theory for our probabilistic logic - for us, this choice was governed by certain model-theoretic facts

2. Formalize our meta-theory - for us, this could be achieved using a first-order theory with equality.

3. Provide a translation of our probabilistic logic into our meta-theory - for us this was achieved by the map $()^*$ .

4. Note that the map $()^*$ translates sentences of our probabilistic logic into quantifier-free formulae and that our meta-theory is formulated by a universal theory.

5. Fix attention on a Gentzen-style axiomatization of the first-order predicate calculus.

6. By Herbrand's theorem we have that our universal axiomatization of our meta-theory can be replaced by a deductively equivalent set of quantifier-free formulae - for us this means that these quantifier-free formulae may be translated back into our logic using the map $()^*$ .

7. By subformula property we may drop use of quantifier rules - this is of no consequence to us since we are only interested in deriving quantifier-free formulae.

8. Using the map $()^*$ we may now reinterpret our rules and quantifier-free axioms for the meta-theory in our probabilistic logic.

9. Finally, we note that by adding in additional probability axioms to our probabilistic logic we may achieve our desired soundness and completeness result.

From the above outline we see that our method has the following immediate limitations;

- our meta-theory must be first-order
- our meta-theory must have equality - this is forced upon us by our model-theoretic requirements
- we need a suitable, well-behaved translation from our logic to our meta-theory - this is necessary so that we may utilize and translate properties of our meta-theory to our logic
- our logic must be translated into a quantifier-free fragment of our meta-theory
- our meta-theory must have a universal axiomatization

These last two points may be overcome by using formulae of a restricted logical complexity (cf. generalizations of existentially-closed structures). Note that this does not affect the validity of Herbrand's theorem and that such a restriction would necessitate the need to place restrictions on the use of quantifier rules rather than drop their use altogether. Again the subformula property ensures things proceed alright. Finally, note that this generalization forces us to consider non-propositional logics. These considerations draw, in particular, our attention to three main limitations with our method. The need for the presence of equality, a first-order theory, and more importantly, a good translation between our logic and meta-theory.

However, by proving our soundness and completeness result in the manner indicated in this paper we have opened up the possibility of transferring known properties from the underlying model and proof theories to our logic of interest. Transferring properties of our model theory to our logic of interest is something that has often been exploited by other authors to obtain definability or complexity information about the logic. However, we have here also opened up the possibility of transferring properties of our proof theory (eg. resolution, unification etc.) to our logic of interest.

## References


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