Pavelkan Logics on the Unit Interval: Proof Boundaries and Sound and Complete Logics

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Abstract

In [6], PAVELKA gives a sound and complete axiomatisation for a logic that reasons with *fuzzy* sets of hypothesis and conclusions rather than just sets, as in (say) classical and intuitionistic logics.

This paper presents axiomatizations that are not only simpler than those presented in [6], but which also have simpler soundness and completeness arguments. In addition, we shall also be concerned with studying PAVELKA's notions of soundness and completeness.

PAVELKA determines soundness and completeness using a limit concept. More precisely, for any fuzzy set X and logical sentence θ:

$$\begin{aligned} &\bigvee\{a \in [0,1] | \text{ from } X \text{ we may derive } \theta \text{ to degree } a\} \\ &= \bigvee\{a \in [0,1] | \text{ for any valuation } W \geq X \cdot W(\theta) \geq a\} \end{aligned}$$

Since we are showing that limit values are equal in our P-soundness and P-completeness results, it should be no surprise that the associated deductive system may fail to derive the limit value, but still be able to derive all non-limit values (ie. those *belief* values strictly less than the limit value). This observation leads us to present a number of logics for which we may characterize the provable limit values.

In particular, we present a logic in which one may assume mutually inconsistent information, but one may not derive every sentence to every *belief* value!

1 Introduction

This paper is based on the ideas first presented in [4], [5] and [6]. Like these papers our semantics shall concentrate on the Łukasiewicz residuated lattice (see [5]).

One of the ideas of PAVELKA's work was to introduce the idea of reasoning not just with logical sentences, but also to carry along with each sentence a lower bound on the degree of *belief* associated with the logical sentence. This leads us to use not sets, but *fuzzy* sets for our hypotheses and conclusions in proofs. PAVELKA then goes on in [6] to present a sound and complete axiomatization based on these ideas.

This paper presents axiomatizations that are not only simpler than those presented in [6], but which also have simpler soundness and completeness arguments. However, this paper is not just concerned with simplifying the work of [4], [5] and [6]. We shall also be concerned with studying the notions of soundness and completeness defined in these papers.

PAVELKA determines soundness and completeness using a limit concept. More precisely, for any fuzzy set X and logical sentence θ:

$$\begin{array}{c} \bigvee\{a \in [0,1] | \text{ from } X \text{ we may derive } \theta \text{ to degree } a\} \\ = \bigvee\{a \in [0,1] | \text{ for any valuation } W \geq X \cdot W(\theta) \geq a\} \end{array}$$

When this is true of a given deductive system we shall say the system is *P-sound* and *P-complete*.

Since we are showing that limit values are equal in our P-soundness and P-completeness results, it should be no surprise that the associated deductive system may fail to derive the limit value, but still be able to derive all non-limit values (ie. those *belief* values strictly less than the limit value). This observation is the basis for much of what follows.

This paper defines two notions of proofs for these axiomatizations.

- A finite P-proof of a logical sentence θ to degree a from fuzzy hypothesis X is any proof of θ to degree a from X.
- A limit P-proof of a logical sentence θ to degree a from fuzzy hypothesis X is any collection of finite P-proofs of θ to degree b from X, for each $b < a$.

In addition, we define the concept of a proof boundary as follows:

$$\begin{array}{c} \mathcal{B} \text{ is a function that given a fuzzy set returns the fuzzy set,} \\ \mathcal{B}(X)(\theta) = \bigvee\{a \in [0,1] | \text{ for any valuation } W \geq X \cdot W(\theta) \geq a\} \end{array}$$

The essence of P-soundness and P-completeness is that any value strictly below the proof boundary is guaranteed to have a finite P-proof, while the boundary value is only guaranteed to have a limit P-proof. In this context, the work of PAVELKA has not concentrated on showing which boundary values have finite P-proofs. This paper sets about addressing this defect.

Section 2 defines all the basic concepts to be used within this paper.

In section 3, we shall be interested in determining if there are any axiomatizations that are P-sound and P-complete, and in addition have no finite P-proofs of boundary sentences. This is answered in the affirmative. An interesting consequence of this soundness and completeness result, is that we will present a logic in which one may assume mutually inconsistent information, but one may not derive every logical sentence to every *belief* value!

eg. $X \not\vdash_S \langle \underline{0}, 1 \rangle$, for any fuzzy set X such that there is no valuation $W \geq X$

Following this, in section 4, we conclude by presenting an axiomatization that is P-sound and P-complete, and in addition has finite P-proofs for boundary sentences based on finite sets of fuzzy hypotheses. [4] shows that this is the best we may hope for. In addition, we comment on the relationship between the work detailed here and in the related works of [3], [4], [5] and [6].

The axiomatization of section 3 and its proof of P-soundness and P-completeness is based on the work presented in [3].

2 Pavelkan Logics, Proof Boundaries and Limit P-Proofs

For $n \in$, $a \in [0,1]$ and $\theta, \phi, \psi \in S$, S is the set of sentences defined by the following grammar:

$$\theta ::= p_n | \underline{a} | \neg\phi | \phi \otimes \psi$$

p_n (for $n \in$) are termed propositions (often simply denoted by p, q, etc). In addition, for $\theta, \phi \in S$, we define $\theta \longrightarrow \phi$ to be $\neg(\theta \otimes \neg\phi)$, $\theta \vee \phi$ to be $(\theta \longrightarrow \phi) \longrightarrow \phi$ and $\theta \wedge \phi$ to be $\neg(\neg\theta \vee \neg\phi)$. The usual bracketing conventions are assumed.

A $[0,1]$-subset is any function $X : S \longrightarrow [0,1]$ and has cardinality $\#\{\theta | X(\theta) > 0\}$. A $[0,1]$-subset is finite if it has finite cardinality and the set of all $[0,1]$-subsets is denoted by $[0,1]^{S\mathsf{L}}$. A finite $[0,1]$-subset X shall be abbreviated by the set:

$$\{\langle \theta_1, a_1 \rangle, \ldots, \langle \theta_n, a_n \rangle\}$$

where the cardinality of X is n and $X(\theta_1) = a_1 > 0, \ldots, X(\theta_n) = a_n > 0$. As in [4], we define:

- $X \geq Y$ iff for any $\theta \in S, X(\theta) \geq Y(\theta)$
- $(X \vee Y)(\theta) = max(X(\theta), Y(\theta)) = X(\theta) \vee Y(\theta)$, for any $\theta \in S$
- $(X \wedge Y)(\theta) = min(X(\theta), Y(\theta)) = X(\theta) \wedge Y(\theta)$, for any $\theta \in S$

Definition 1 *A P-sequent is a pair $X|\langle\theta, a\rangle$ where X is a finite $[0,1]$-subset, $\theta \in S$ and $a \in [0,1]$.*

We shall often abbreviate the P-sequents $X \vee \{\langle\theta_1, a_1\rangle, \ldots, \langle\theta_n, a_n\rangle\}|\langle\phi, b\rangle$ and $X \vee Y|\langle\theta, a\rangle$ by $X, \langle\theta_1, a_1\rangle, \ldots, \langle\theta_n, a_n\rangle|\langle\phi, b\rangle$ and $X, Y|\langle\theta, a\rangle$ respectively.

Definition 2 *An n-ary (for $n \geq 0$) P-rule is of the form:*

$$\frac{X_1|\langle\theta_1,a_1\rangle;\ldots;X_n|\langle\theta_n,a_n\rangle}{X|\langle\theta,a\rangle}[f_1, \ldots, f_{1+\#X}]$$

where $X_1, \ldots, X_n, X \in [0,1]^{S\mathsf{L}}$ are all finite, $\theta_1, \ldots, \theta_n, \theta \in S$, $a_1, \ldots, a_n, a \in [0,1]$, $f_1, \ldots, f_{1+\#X}$ are fixed partial functions from $([0,1]^{S\mathsf{L}} \times S \times [0,1])^n$ to $S \times [0,1]$ such that whenever $f_1, \ldots, f_{1+\#X}$ are defined:

- $(\theta, a) = f_{1+\#X}((X_1, \theta_1, a_1), \ldots, (X_n, \theta_n, a_n))$
- *for all $\phi \in S$,*

$$Y(\phi) = \bigvee_{i=1}^{\#X}\{snd\ f_i((X_1, \theta_1, a_1), \ldots, (X_n, \theta_n, a_n)) \\ |fst\ f_i((X_1, \theta_1, a_1), \ldots, (X_n, \theta_n, a_n)) = \phi\}$$

- $f_1, \ldots, f_{1+\#X}$ *are all monotone (where for $X, Y \in [0,1]^{S\mathsf{L}}$, $\theta \in S$ and $a, b \in [0,1]$):*

$$(X, \theta, a) \leq (Y, \theta, b) \text{ iff } X \leq Y \text{ and } a \leq b)$$

Notice that here and elsewhere, we set $\bigvee \emptyset = 0$.

P-axioms are simply zero-ary P-rules. In writing down a P-rule we do not make explicit reference to the functions $f_1, \ldots, f_{1+\#X}$. Rather we shall write down the P-rule in such a way that these partial functions are clear from the P-rules context.

A P-axiomatization is simply a collection of P-rules (and P-axioms).

A P-proof is defined as usual once a P-axiomatization has been fixed.

Given a P-axiomatization and its associated provability predicate $\vdash_P$, by:

$$X \vdash_P \langle \theta, a \rangle$$

for $X \in [0,1]^{S\mathsf{L}}$, $\theta \in S$ and $a \in [0,1]$, we shall mean:

$$\text{there is a finite } X_0 \leq X \text{ such that } \vdash_P X_0 | \langle \theta, a \rangle$$

Definition 3 *A $W \in [0,1]^{S\mathsf{L}}$ is said to be a valuation if the following hold, for any $\theta, \phi \in S$ and $a \in [0,1]$:*

- $W(\underline{a}) = a$
- $W(\neg\theta) = 1 - W(\theta)$
- $W(\theta \otimes \phi) = max(0, W(\theta) + W(\phi) - 1)$

Notice that the following are all easy consequences of the above:

- $W(\theta \vee \phi) = max(W(\theta), W(\phi))$
- $W(\theta \wedge \phi) = min(W(\theta), W(\phi))$
- $W(\theta \longrightarrow \phi) = min(0, 1 - W(\theta) + W(\phi))$

Definition 4 *By $X \models \langle \theta, a \rangle$ (for $X \in [0,1]^{S\mathsf{L}}$, $\theta \in S$ and $a \in [0,1]$), we shall mean that:*

$$\textit{for all valuations } W, W \geq X \textit{ implies } W(\theta) \geq a$$

Definition 5 *A P-axiomatization $\vdash_P$ is P-sound if, for all $X \in [0,1]^{S\mathsf{L}}$ and $\theta \in S$:*

$$\bigvee\{a \in [0,1] | X \vdash_P \langle \theta, a \rangle\} \leq \bigvee\{a \in [0,1] | X \models \langle \theta, a \rangle\}$$

Definition 6 *A P-axiomatization $\vdash_P$ is P-complete if, for all $X \in [0,1]^{S\mathsf{L}}$, $\theta \in S$ and $a \in [0,1]$:*

$$\bigvee\{a \in [0,1] | X \vdash_P \langle \theta, a \rangle\} \geq \bigvee\{a \in [0,1] | X \models \langle \theta, a \rangle\}$$

Definition 7 *For any $X \in [0,1]^{S\mathsf{L}}$ the P-proof boundary of X is the $[0,1]$-subset, $\mathcal{B}(X) \in [0,1]^{S\mathsf{L}}$ such that:*

$$\textit{for all } \theta \in S,$$
$$\mathcal{B}(X)(\theta) = \bigvee\{a \in [0,1] | \textit{ for any valuation } W \geq X \cdot W(\theta) \geq a\}$$

Theorem 1 *Let $\vdash_P$ be the provability predicate for some P-axiomatization. Then:*

- *$\vdash_P$ is P-sound iff for any $X \in [0,1]^{S\text{Ł}}$ and $\theta \in S$,*

$$\bigvee\{a \in [0,1] | X \vdash_P \langle\theta, a\rangle\} \leq \mathcal{B}(X)(\theta)$$

- *$\vdash_P$ is P-complete iff for any $X \in [0,1]^{S\text{Ł}}$ and $\theta \in S$,*

$$\bigvee\{a \in [0,1] | X \vdash_P \langle\theta, a\rangle\} \geq \mathcal{B}(X)(\theta)$$

Proof. Follows easily by definition of the proof boundary for X. ■

Definition 8 *Let $\vdash_P$ be the provability predicate for some P-axiomatization, $X \in [0,1]^{S\text{Ł}}$, $\theta \in S$ and $a \in [0,1]$. Then:*

- *we have a finite P-proof (in $\vdash_P$) of $\langle\theta, a\rangle$ from X iff $X \vdash_P \langle\theta, a\rangle$*
- *we have a limit P-proof (in $\vdash_P$) of $\langle\theta, a\rangle$ from X iff for any $b < a$, we have a finite P-proof (in $\vdash_P$) of $\langle\theta, a\rangle$ from X*

3 The Minimal Pavelkan Axiomatization

Definition 9 *Let $\vdash_S$ be the provability predicate for the following axiomatization:*

$$\frac{}{|\langle\underline{a},b\rangle} \ (for\ a > b) \qquad \frac{}{\langle\underline{a},b\rangle|\langle\theta,c\rangle} \ (for\ a < b\ and\ c < 1)$$

$$\frac{}{Y|\langle\theta,a\rangle} \ (for\ Y(\theta) > a) \qquad \frac{}{\langle\theta\otimes\neg\theta,a\rangle|\langle\underline{0},c\rangle} \ (for\ a > 0\ and\ c < 1)$$

$$\frac{}{|\langle\theta\longrightarrow\theta,a\rangle} \ (for\ a < 1)$$

$$\frac{Y_1|\langle\theta,a\rangle; Y_2,\langle\theta,a\rangle|\langle\phi,b\rangle}{Y_1,Y_2|\langle\phi,b\rangle} cut$$

$$\frac{Y_1|\langle\theta,a\rangle; Y_2|\langle\phi,b\rangle; Y_3|\langle\underline{a\otimes b},0\rangle}{Y_1,Y_2,Y_3|\langle\theta\otimes\phi,a\otimes b\rangle} \otimes -in$$

$$\frac{Y_1,\langle\neg\theta,a\rangle|\langle\phi,b\rangle; Y_2,\langle\psi,c\rangle|\langle\phi,b\rangle}{Y_1,Y_2,\langle\underline{0},a\otimes c\rangle,\langle\theta\longrightarrow\psi,\neg a\longrightarrow c\rangle|\langle\phi,b\rangle} C1 \qquad \frac{Y_1,\langle\theta,a\rangle|\langle\phi,b\rangle; Y_2,\langle\psi,c\rangle|\langle\phi,b\rangle}{Y_1,Y_2,\langle\theta\otimes\psi,(a\otimes c)\vee d\rangle|\langle\phi,b\rangle} C2(d)$$

$$\frac{Y,\langle\theta\longrightarrow\theta,1\rangle|\langle\phi,b\rangle}{Y|\langle\phi,b\rangle} tout$$

where $a, b, c, d \in [0,1]$ *with* $d \neq 0$*,* $\theta, \phi, \psi \in S$ *and* $Y, Y_1, Y_2, Y_3 \in [0,1]^{S\mathsf{L}}$ *are all finite.*

An easy inspection of the above definition demonstrates that the rules for $\vdash_S$ are in fact all monotone, and so we have indeed presented a P-axiomatization for $\vdash_S$.

Definition 10 *For any* $X \in [0,1]^{S\mathsf{L}}$*,* $\theta \in S$ *and* $a \in [0,1]$*:*

$$X \models_S \langle \theta, a \rangle \text{ iff } a \neq 1 \text{ and for any valuation } W \geq X,\ W(\theta) > a$$

Definition 11 *Let* $X \in [0,1]^{S\mathsf{L}}$ *then:*

- X *is limit inconsistent iff for any* $\theta \in S$ *and* $a \in [0,1)$*,* $X \vdash_S \langle \theta, a \rangle$
- X *is limit consistent iff* X *is not limit inconsistent*

We now go on to demonstrate that the finite P-proofs of $\vdash_S$ are precisely those validated by $\models_S$.

Theorem 2 *For any* $X \in [0,1]^{S\mathsf{L}}$*,* $\theta \in S$ *and* $a \in [0,1]$*:*

$$X \vdash_S \langle \theta, a \rangle \text{ implies } X \models_S \langle \theta, a \rangle$$

Proof. Straightforward proof by induction on the number of P-rules occurring in a P-proof of the P-sequent $X|\langle \theta, a \rangle$, for finite $X \in [0,1]^{S\mathsf{L}}$.

We simply outline a number of particular steps in the inductive argument.

$\otimes - in$ rule: Our P-proof ends in an application of the rule $\otimes - in$. So we have P-proofs of $Y_1|\langle \phi_1, b_1 \rangle$, $Y_2|\langle \phi_2, b_2 \rangle$ and $Y_3|\langle \underline{b_1 \otimes b_2}, 0 \rangle$. By our inductive hypothesis we then have that:

$$Y_1 \models_S \langle \phi_1, b_1 \rangle,\ Y_2 \models_S \langle \phi_2, b_2 \rangle \text{ and } Y_3 \models_S \langle \underline{b_1 \otimes b_2}, 0 \rangle$$

Now let W be a valuation such that, $W \geq Y_1 \vee Y_2 \vee Y_3$. Then:

$$W(\phi_1) > b_1 \neq 1,\ W(\phi_2) > b_2 \neq 1 \text{ and } W(\underline{b_1 \otimes b_2}) > 0 \neq 1$$

Thus:

$$0 < W(\underline{b_1 \otimes b_2}) = b_1 \otimes b_2 = max(0, b_1 + b_2 - 1) = b_1 + b_2 - 1$$

and:

$$W(\phi_1) + W(\phi_2) - 1 > b_1 + b_2 - 1 > 0$$

Hence:

$$W(\phi_1 \otimes \phi_2) = max(0, W(\phi_1) + W(\phi_2) - 1) = W(\phi_1) + W(\phi_2) - 1$$

Thus we have that:

$$\text{for any valuation } W \geq Y_1 \vee Y_2 \vee Y_3,\ W(\phi_1 \otimes \phi_2) > b_1 \otimes b_2$$

To obtain our required result for this case we now only need to verify that $b_1 \otimes b_2 \neq 1$. Since $b_1, b_2 < 1$ we have that:

$$b_1 + b_2 - 1 < 1$$

and so:

$$1 > max(0, b_1 + b_2 - 1) = b_1 \otimes b_2$$

as we required.

$C1$-rule: Our P-proof ends in an application of the rule $C1$. So we have P-proofs of $Y_1, \langle \neg\theta, a \rangle | \langle \phi, b \rangle$ and $Y_2, \langle \psi, c \rangle | \langle \phi, b \rangle$. By our inductive hypothesis we then have that:

$$Y_1, \langle \neg\theta, a \rangle \models_S \langle \phi, b \rangle \text{ and } Y_2, \langle \psi, c \rangle \models_S \langle \phi, b \rangle$$

Trivially then $b \neq 1$. So let W be a valuation such that, $W \geq Y_1 \vee Y_2 \vee \langle \underline{0}, a \otimes c \rangle \vee \langle \theta \longrightarrow \psi, \neg a \longrightarrow c \rangle$. Then:

$$W(\underline{0}) = 0 \geq a \otimes c = max(0, a + c - 1) \geq 0$$

Thus:

$$a + c \leq 1$$

and so:

$$W(\theta \longrightarrow \psi) \geq \neg a \longrightarrow c = min(1, a + c) = a + c$$

Assume now that $W(\neg\theta) = 1 - W(\theta) < a$ and $W(\psi) < c$ and argue for a contradiction. Now:

$$W(\theta \longrightarrow \psi) = min(1, 1 - W(\theta) + W(\psi)) \text{ and}$$
$$1 - W(\theta) + W(\psi) < a + c \leq 1$$

Thus, we have our required contradiction since:

$$a + c \leq W(\theta \longrightarrow \psi) < a + c$$

Our required result now follows easily using our inductive hypothesis.

$C2(d)$-rule: Our P-proof ends in an application of the rule $C2(d)$, for some fixed $d > 0$. So we have P-proofs of $Y_1, \langle\theta, a\rangle|\langle\phi, b\rangle$ and $Y_2, \langle\psi, c\rangle|\langle\phi, b\rangle$. By our inductive hypothesis we then have that:

$$Y_1, \langle\theta, a\rangle \models_S \langle\phi, b\rangle \text{ and } Y_2, \langle\psi, c\rangle \models_S \langle\phi, b\rangle$$

Trivially then $b \neq 1$. So let W be a valuation such that, $W \geq Y_1 \vee Y_2 \vee \langle\theta \otimes \psi, (a \otimes c) \vee d\rangle$. Then:

$$W(\theta \otimes \psi) \geq a \otimes c \text{ and } W(\theta \otimes \psi) = max(0, W(\theta) + W(\psi) - 1) \geq d > 0$$

Thus:

$$W(\theta \otimes \psi) = W(\theta) + W(\psi) - 1$$

Assume now that $W(\theta) < a$ and $W(\psi) < c$ and argue for a contradiction. But:

$$0 < W(\theta \otimes \psi) = W(\theta) + W(\psi) - 1 < a + c - 1 = a \otimes c$$

and so we have our required contradiction. Our required result now follows easily using our inductive hypothesis.

■

Before demonstrating the completeness of $\vdash_S$ it is first necessary to verify a series of lemmata.

Lemma 3 *Let* $X \in [0,1]^{S\mathsf{L}}$ *then,*

- X *is limit inconsistent iff* $X \vdash_S \langle \underline{0}, \frac{1}{2} \rangle$
- *if* $\langle \theta, a \rangle \in S \times [0,1)$ *and* $X \nvdash_S \langle \theta, a \rangle$ *then for any* $b \in (a,1]$, $X \vee \langle \neg\theta, \neg b \rangle$ *is limit consistent*

Proof.

- Easily verified.
- Argue by contradiction and so assume that there is a $b \in (a,1]$ for which $X \vee \langle \neg\theta, \neg b \rangle$ is limit inconsistent. Thus in particular we have, by definition of limit inconsistency, that:

  $$X, \langle \neg\theta, \neg b \rangle \vdash_S \langle \theta, a \rangle$$

  Now let $c = \frac{a+b}{2}$ then clearly, $a < c < b$ and so by C1 and the axiom instance $\langle \theta, c \rangle | \langle \theta, a \rangle$ we get that:

  $$X, \langle \underline{0}, \neg b \otimes c \rangle, \langle \theta \longrightarrow \theta, b \longrightarrow c \rangle \vdash_S \langle \theta, a \rangle$$

  Thus by cut and the axiom instance $|\langle \theta \longrightarrow \theta, b \longrightarrow c \rangle$ (which is valid by choice of c and since $b \leq c$ iff $b \longrightarrow c = 1$) we get that:

  $$X, \langle \underline{0}, \neg b \otimes c \rangle \vdash_S \langle \theta, a \rangle$$

  By monotonicity of $\otimes$ we have that:

  $$0 \leq \neg b \otimes c \leq \neg b \otimes b = max(0, \neg b + b - 1) = max(0,0) = 0$$

  and so our required contradiction since:

$$X \vee \langle \underline{0}, \neg b \otimes c \rangle = X \vee \langle \underline{0}, 0 \rangle = X$$

and so:

$$X \vdash_S \langle \theta, a \rangle$$

■

Lemma 4 (An approximation to Lindenbaum's Lemma) *Let $X \in [0,1]^{S\mathsf{L}}$ be limit consistent. Then there is a maximal collection of $[0,1]$-subsets, $\mathcal{S}_X$ say, satisfying the following conditions:*

- $X \in \mathcal{S}_X$
- *for any $S_1, \ldots, S_m \in \mathcal{S}_X$, $S_1 \vee \ldots \vee S_m$ is limit consistent*

Proof. [Assuming Zorn's Lemma] Let $\mathcal{C}$ be the family of all sets satisfying both the conditions above.

Clearly $\{X\}$ satisfies both of these conditions and so $\mathcal{C} \neq \emptyset$. Obviously $\mathcal{C}$ is a poset under the usual subset relation.

Now let $C \subseteq \mathcal{C}$ be a chain. If $C = \emptyset$ then any member of $\mathcal{C}$ will suffice as an upper bound to C. So assume that $C \neq \emptyset$ and consider the set $\cup C$. Clearly if $\cup C \in \mathcal{C}$ then we have an upper bound to C and so by Zorn's lemma $\mathcal{C}$ has a maximal element, $\mathcal{S}_X$ say. Trivially $\mathcal{S}_X$ satisfies both the conditions above and so we obtain our required result.

It remains to establish that $\cup C \in \mathcal{C}$. In order to do this it suffices to show that $\cup C$ satisfies both the above conditions.

As every member of C satisfies the first condition it is immediate that $\cup C$ also satisfies this condition.

Let $S_1, \ldots, S_m \in \cup C$. Since C is a chain of sets ordered by $\subseteq$, the members of C form a cumulative collection and so, there exists an $S \in C \subseteq \mathcal{C}$ such that:

$$S_1, \ldots, S_m \in S$$

Hence, since $S \in \mathcal{C}$ and every member of $\mathcal{C}$ satisfies both the conditions above, $S_1 \vee \ldots \vee S_m$ is limit consistent and so $\cup C$ satisfies the second condition as required. ■

A straightforward consequence of lemma 4 is that for any $Y \in [0,1]^{S\mathsf{L}}$:

$$\forall S \in \mathcal{S}_X \cdot S \vee Y \text{ is limit consistent implies } Y \in \mathcal{S}_X$$

That this holds can be seen from the following:

Firstly, we show this for $Y = S_1 \vee S_2$, where $S_1, S_2 \in \mathcal{S}_X$.

Clearly $\mathcal{S}_X \subseteq \mathcal{S}_X \cup \{S_1 \vee S_2\}$. So to verify membership it suffices to show that inclusion holds in the other direction. To do this we need only verify that $\mathcal{S}_X \cup \{S_1 \vee S_2\}$ satisfies both the conditions of the previous lemma.

Trivially, $X \in \mathcal{S}_X \cup \{S_1 \vee S_2\}$.

Let $T_1, \ldots, T_m \in \mathcal{S}_X \cup \{S_1 \vee S_2\}$. Then trivially, $T_1 \vee \ldots \vee T_m \vee S_1 \vee S_2$ is limit consistent as required.

We now argue in a similar fashion to verify our original claim. As above we need only verify that $\mathcal{S}_X \cup \{Y\}$ satisfies both the conditions of lemma 4.

Clearly, $X \in \mathcal{S}_X \cup \{Y\}$.

So let $S_1, \ldots, S_m \in \mathcal{S}_X \cup \{Y\}$ be distinct. If $Y \notin \{S_1, \ldots, S_m\}$ then trivially $S_1 \vee \ldots \vee S_m$ is limit consistent. If $Y \in \{S_1, \ldots, S_m\}$, wlog $Y = S_m$ say, then $S_1, \ldots, S_{m-1} \in \mathcal{S}_X$ and so by an obvious and easy extension of the above we have that $S_1 \vee \ldots \vee S_{m-1} \in \mathcal{S}_X$. Thus by our original hypothesis we have that $S_1 \vee \ldots \vee S_m$ is limit consistent as we required.

Lemma 5 *Let $X \in [0,1]^{S\mathsf{L}}$ be limit consistent and $\mathcal{S}_X$ be the maximal collection of $[0,1]$-subsets guaranteed by the previous lemma. Define W_X to be the $[0,1]$-subset such that for any $\theta \in S$:*

$$W_X(\theta) = \bigvee\{b \in [0,1] | \exists Y \in \mathcal{S}_X \cdot Y \vdash_S \langle \theta, b \rangle\}$$

Then $W_X \geq X$ and W_X is a valuation.

Proof. That $W_X \geq X$ follows immediately from the definition of W_X and $\vdash_S$. Next note that by instances of $Y|\langle \theta, b \rangle$, for $b < Y(\theta)$, and the definition of W_X we have that for any $\theta \in S$, if $a \in [0, W_X(\theta))$ then there is a $Y \in \mathcal{S}_X$ such that:

$$Y \vdash_S \langle \theta, a \rangle$$

This fact is used below without any further mention.

Now in order to show that W_X is a valuation it suffices to show that W_X satisfies the following conditions:

- for any $a \in [0,1]$, $W_X(\underline{a}) = a$
- for any $\theta \in S$, $W_X(\neg\theta) = 1 - W_X(\theta)$

- for any $\theta, \phi \in S$, $W_X(\theta \otimes \phi) = max(0, W_X(\theta) + W_X(\phi) - 1)$

These each follow from the following arguments:

- Let $a \in [0,1]$. If $a > 0$ then let $b \in [0, a)$ and so by the axiom instance $|\langle \underline{a}, b\rangle$ we have that for any $S \in \mathcal{S}_X$:

$$S \vdash_S \langle \underline{a}, b\rangle$$

  Hence by the definition of W_X we have $W_X(\underline{a}) \geq a$. If $a = 0$ this is trivially true. Now assume, for a contradiction, that $W_X(\underline{a}) > a$. Now let $c \in (a, W_X(\underline{a}))$ and $S \in \mathcal{S}_X$ be such that:

$$S \vdash_S \langle \underline{a}, c\rangle$$

  then by cut and the axiom instance $\langle \underline{a}, c\rangle | \langle \underline{0}, \frac{1}{2}\rangle$ we get that:

$$S \vdash_S \langle \underline{0}, \tfrac{1}{2}\rangle$$

  and so by lemma 3 that S is limit inconsistent. But $S \in \mathcal{S}_X$ and so by definition of $\mathcal{S}_X$ is also limit consistent. Thus we have our required contradiction.

- Let $\theta \in S$ and argue by contradiction. Thus either:

  a: $\neg W_X(\theta) > W_X(\neg\theta)$
  b: $\neg W_X(\theta) < W_X(\neg\theta)$

  a: let $x \in (W_X(\neg\theta), \neg W_X(\theta))$ then, using the comments preceding lemma 5, we have $S_1, S_2 \in \mathcal{S}_X$ such that:

$$S_1 \vee \langle \neg\theta, x\rangle \text{ and } S_2 \vee \langle \theta, \neg x\rangle \text{ are limit inconsistent}$$

  For, if $\forall S \in \mathcal{S}_X \cdot S \vee \langle \neg\theta, x\rangle$ is limit consistent then, using the comments preceding lemma 5, we have $\langle \neg\theta, x\rangle \in \mathcal{S}_X$ and so by definition of W_X and the choice of x that:

$$x > W_X(\neg\theta) \geq x$$

  and so a contradiction. Thus we have the existence of $S_1 \in \mathcal{S}_X$. Similarly, we may argue for the existence of the required $S_2 \in \mathcal{S}_X$. Now by lemma 3:

$$S_1, \langle \neg\theta, x\rangle \vdash_S \langle \underline{0}, \tfrac{1}{2}\rangle \text{ and } S_2, \langle \theta, \neg x\rangle \vdash_S \langle \underline{0}, \tfrac{1}{2}\rangle$$

So by C1 and since $\neg x \longrightarrow \neg x = 1$, we get:

$$S_1, S_2, \langle \underline{0}, x \otimes \neg x\rangle, \langle \theta \longrightarrow \theta, 1\rangle \vdash_S \langle \underline{0}, \tfrac{1}{2}\rangle$$

and by tout and since:

$$S_1 \vee S_2 \vee \langle \underline{0}, x \otimes \neg x\rangle = S_1 \vee S_2 \vee \langle \underline{0}, 0\rangle = S_1 \vee S_2$$

we now get that:

$$S_1, S_2 \vdash_S \langle \underline{0}, \tfrac{1}{2}\rangle$$

and so by lemma 3, $S_1 \vee S_2$ is limit inconsistent. But $S_1, S_2 \in \mathcal{S}_X$ and so $S_1 \vee S_2$ is also limit consistent. Thus we have our required contradiction.

b: let $x \in (\neg W_X(\theta), W_X(\neg\theta))$, $y \in (x, W_X(\neg\theta))$ and $z \in (1 - x, W_X(\theta))$. Then:

$$y + z > x + 1 - x = 1$$

and so:

$$0 < y + z - 1 = max(0, y + z - 1) = y \otimes z$$

Now since $y < W_X(\neg\theta)$ and $z < W_X(\theta)$ it follows that there exists $S_1, S_2 \in \mathcal{S}_X$ such that:

$$S_1 \vdash_S \langle \neg\theta, y\rangle \text{ and } S_2 \vdash_S \langle \theta, z\rangle$$

Thus by $\otimes - in$ and the axiom instance (which is valid by choice of y and z) $|\langle \underline{y \otimes z}, 0\rangle$ we get that:

$$S_1, S_2 \vdash_S \langle \neg\theta \otimes \theta, y \otimes z\rangle$$

and so by cut and the axiom instance $\langle \neg\theta \otimes \theta, y \otimes z\rangle | \langle \underline{0}, \tfrac{1}{2}\rangle$ we get that:

$$S_1, S_2 \vdash_S \langle \underline{0}, \tfrac{1}{2}\rangle$$

and so by lemma 3, $S_1 \vee S_2$ is limit inconsistent. But $S_1, S_2 \in \mathcal{S}_X$ and so $S_1 \vee S_2$ is also limit consistent. Hence we have our required contradiction.

- Let $\theta, \phi \in S$. Now for any $x \in [0, W_X(\theta) \otimes W_X(\phi))$ we have $y < W_X(\theta)$ and $z < W_X(\phi)$ such that:

$$0 \leq x < y \otimes z$$

Hence, we have $S_1, S_2 \in \mathcal{S}_X$ such that:

$$S_1 \vdash_S \langle \theta, y \rangle, \; S_2 \vdash_S \langle \phi, z \rangle \text{ and } |\langle \underline{y \otimes z}, 0 \rangle$$

Hence by $\otimes - in$:

$$S_1, S_2 \vdash_S \langle \theta \otimes \phi, y \otimes z \rangle$$

and so by cut and the axiom $\langle \theta \otimes \phi, y \otimes z \rangle | \langle \theta \otimes \phi, x \rangle$:

$$S_1, S_2 \vdash_S \langle \theta \otimes \phi, x \rangle$$

But by our remarks preceding lemma 5, $S_1 \vee S_2 \in \mathcal{S}_X$, and so, $x \leq W_X(\theta \otimes \phi)$.

$$\text{ie. } W_X(\theta) \otimes W_X(\phi) \leq W_X(\theta \otimes \phi)$$

Now let $x \in (W_X(\theta) \otimes W_X(\phi), W_X(\theta \otimes \phi))$ then we have $y > W_X(\theta)$ and $z > W_X(\phi)$ such that:

$$0 < y \otimes z < x$$

So we have an $S_1 \in \mathcal{S}_X$ such that:

$$S_1 \vdash_S \langle \theta \otimes \phi, x \rangle$$

and so by cut and the axiom $\langle \theta \otimes \phi, x \rangle | \langle \theta \otimes \phi, y \otimes z \rangle$ that:

$$S_1 \vdash_S \langle \theta \otimes \phi, y \otimes z \rangle$$

Also, we have $S_2, S_3 \in \mathcal{S}_X$ such that:

$$S_2 \vee \langle \theta, y \rangle \text{ and } S_3 \vee \langle \phi, z \rangle \text{ are limit inconsistent}$$

Hence, by lemma 3, $C2(y \otimes z)$ and since $|\langle \underline{y \otimes z}, 0 \rangle$ we have that:

$$S_2, S_3, \langle \theta \otimes \phi, y \otimes z \rangle \vdash_S \langle \underline{0}, \tfrac{1}{2} \rangle$$

a use of cut now yields:

$$S_1, S_2, S_3 \vdash_S \langle \underline{0}, \tfrac{1}{2} \rangle$$

and so by lemma 3 our required contradiction.

■

Theorem 6 *For any $X \in [0,1]^{S\mathsf{L}}$, $\theta \in S$ and $a \in [0,1]$:*

$$X \models_S \langle \theta, a \rangle \textit{ implies } X \vdash_S \langle \theta, a \rangle$$

Proof. Argue by contradiction and so assume that we have an $X \in [0,1]^{S\mathsf{L}}$, $\theta \in S$ and $a \in [0,1]$ such that:

$$X \models_S \langle \theta, a \rangle \text{ and } X \not\vdash_S \langle \theta, a \rangle$$

If X is limit inconsistent we have that:

$$X \vdash_S \langle \theta, b \rangle, \text{ for any } b < 1$$

Thus $a = 1$. But then we have a contradiction since:

$$X \models_S \langle \theta, a \rangle$$

and so X must be limit consistent. Since $X \models_S \langle \theta, a \rangle$ it follows that for any valuation $W \geq X$, $W(\theta) > a$. Now since:

$$\mathcal{B}(X)(\theta) = \bigwedge \{ W(\theta) | W \text{ a valuation and } W \geq X \}$$

then:

$$\forall \epsilon > 0 \exists W_\epsilon \geq X \text{ such that } W_\epsilon(\theta) \in (\mathcal{B}(X)(\theta), \mathcal{B}(X)(\theta) - \epsilon) \cap [0,1]$$

and so, by continuity of $\neg$ and $\otimes$, we have a valuation $W' \geq X$ which is such that:

$$W'(\theta) = \mathcal{B}(X)(\theta)$$

and so, by definition of $\models_S$:

$$X \not\models_S \langle \theta, \mathcal{B}(X)(\theta) \rangle$$

Thus, we must have, by definition of $\mathcal{B}(X)(\theta)$ and since $X \models_S \langle \theta, a \rangle$, that:

$$1 \geq \mathcal{B}(X)(\theta) > a$$

Now for any $c \in (a, \mathcal{B}(X)(\theta))$ it follows by lemma 3 that:

$$Y_c = X \vee \langle \neg\theta, \neg c \rangle \text{ is limit consistent}$$

and so by lemma 5 we have a valuation $W_{Y_c} \geq Y_c \geq X$. But:

$$\begin{aligned} \mathcal{B}(X)(\theta) &= \bigvee \{ b \in [0,1] | X \models \langle \theta, b \rangle \} \\ &\leq \bigwedge \{ W''(\theta) | W'' \text{ a valuation and } W'' \geq X \} \end{aligned}$$

Thus, since $W_{Y_c} \geq X$, we have:

$$W_{Y_c}(\theta) \geq \mathcal{B}(X)(\theta) > c$$

But $W_{Y_c} \geq Y_c$ and so we also have:

$$W_{Y_c}(\neg\theta) = \neg W_{Y_c}(\theta) \geq \neg c \text{ and so } W_{Y_c}(\theta) \leq c$$

and our required contradiction. ■

Corollary 7 (P-soundness and P-completeness) *The P-axiomatization* $\vdash_S$ *is P-sound and P-complete*

Proof. Let $X \in [0,1]^{S\mathsf{L}}$ and $\theta \in S$ then by theorems 2 and 6 we get that:

$$\bigvee \{ b \in [0,1] | X \vdash_S \langle \theta, b \rangle \} = \bigvee \{ b \in [0,1] | X \models_S \langle \theta, b \rangle \}$$

Thus by the definition of $\models_S$ and $\models$, we have that:

$$\bigvee \{ b \in [0,1] | X \models_S \langle \theta, b \rangle \} \leq \bigvee \{ b \in [0,1] | X \models \langle \theta, b \rangle \}$$

and so that $\vdash_S$ is P-sound.

To show that $\vdash_S$ is P-complete we argue by contradiction and so assume that:

$$\bigvee\{b \in [0,1] | X \vdash_S \langle\theta, b\rangle\} < \bigvee\{b \in [0,1] | X \models \langle\theta, b\rangle\}$$

Let $c < \bigvee\{b \in [0,1] | X \models \langle\theta, b\rangle\} \leq 1$ then for any valuation $W \geq X$ we must have that $W(\theta) > c$. Hence:

$$X \models_S \langle\theta, c\rangle$$

and so by theorem 6 we have that:

$$X \vdash_S \langle\theta, c\rangle$$

Thus:

$$c \leq \bigvee\{b \in [0,1] | X \vdash_S \langle\theta, b\rangle\} \leq \bigvee\{b \in [0,1] | X \models_S \langle\theta, b\rangle\}$$

and so a contradiction on the choice of c. ■

In addition we have the following interesting corollary:

Corollary 8 *Let $X \in [0,1]^{SL}$ then for any $\langle\theta, a\rangle \in \mathcal{B}(X)$ we have a limit P-proof of $\langle\theta, a\rangle$ from X, but no finite P-proof.*

Proof. Let $X \in [0,1]^{SL}$ and $\langle\theta, a\rangle \in \mathcal{B}(X)$, then either:

i: there is no valuation W such that $W \geq X$ and so trivially for any $\phi \in S$ we have that:

$$X \models \langle\phi, 1\rangle$$

and so that:

$$\mathcal{B}(X) = \{\langle\phi, 1\rangle | \phi \in S\}$$

But then we must have $a = 1$ and so:

$$X \not\models_S \langle\theta, a\rangle$$

But then by theorem 3 we have that:

$$X \not\vdash_S \langle\theta, a\rangle$$

and so there is no finite P-proof of $\langle\theta, a\rangle$ from X in $\vdash_S$. That there is a limit P-proof follows from corollary 7.

ii: there is a valuation W such that $W \geq X$. If $a = 1$ then trivially we get that:

$$X \not\models_S \langle\theta, a\rangle$$

So assume $a < 1$. Then by similar arguments to the proof of theorem 6 we have a $W \geq X$ such that:

$$W(\theta) = a = \mathcal{B}(X)(\theta) < 1$$

and so:

$$X \not\models_S \langle\theta, a\rangle$$

But then in either case, we have by theorem 3:

$$X \not\vdash_S \langle\theta, a\rangle$$

and so there is no finite P-proof of $\langle\theta, a\rangle$ from X in $\vdash_S$. That there is a limit P-proof follows from corollary 7.

■

A particular consequence of this corollary is that $\vdash_S$ allows one to assume inconsistent members of $[0,1]^{S\mathsf{L}}$ (ie. there is no valuation W such that $W \geq X$), without becoming inconsistent (ie. being able to prove every member of $S \times [0,1]$)!

4 Other Pavelkan Axiomatizations

The notion of a P-axiomatization was first defined in [4], [5] and [6]. However, Pavelka's work differs from ours in a number of ways.

Firstly, a P-rule is defined to be as here, except that in addition the fixed partial functions associated with the P-rule must be *upper semi-continuous* (see [4]). We simply note that the P-rules presented in definition 9 may be shown to be *upper semi-continuous* (see [4]).

Next, PAVELKA uses a Hilbert style axiomatization rather than a Gentzen style, as used here and in [3]. The standard translation between these two styles of axiomatization enables us to translate PAVELKA's work into the context of this paper.

Finally, PAVELKA uses a different logical language within which to formulate his P-axiomatizations. However, all PAVELKA's connectives are definable within the logical language presented in this paper.

The proof of P-soundness and P-completeness presented for the P-axiomatization of [6], is considerably more complicated than that of this paper. [3] presents, for a different P-axiomatization, a simplified version of the P-soundness and P-completeness result first presented in [6].

The P-axiomatizations of PAVELKA and PARIS both suffer from the same problem. Namely, they do not attempt to classify which boundary objects have finite P-proofs.

Section 3 has demonstrated that it is possible to answer such questions. In particular, we have seen that it is possible to provide P-axiomatizations for which there are no finite P-proofs on our proof boundaries. This leads us to our next question. Is it possible to provide P-axiomatizations for which there are finite P-proofs for all of the proof boundary? [5] has shown that this can not be the case. Indeed, the best we can hope for is such a result for the proof boundary of finite sets of hypotheses.

In the following we demonstrate that this is achievable.

Definition 12 *Let $\vdash_{\underline{L}}$ be the provability predicate for the following axiomatization:*

$$\frac{}{Y|\langle\theta,a\rangle}\ (for\ Y(\theta)\geq a) \qquad \frac{}{|\langle\underline{a},b\rangle}\ (for\ a\geq b)$$

$$\frac{}{|\langle\neg\underline{a}\longrightarrow\underline{\neg a},1\rangle} \qquad \frac{}{|\langle\underline{a}\otimes\underline{b}\longrightarrow\underline{a\otimes b},1\rangle}$$

$$\frac{}{|\langle\theta\longrightarrow(\phi\longrightarrow\theta),1\rangle} \qquad \frac{}{|\langle(\theta\longrightarrow\phi)\longrightarrow((\phi\longrightarrow\psi)\longrightarrow(\theta\longrightarrow\psi)),1\rangle}$$

$$\frac{}{|\langle(\neg\theta\longrightarrow\neg\phi)\longrightarrow(\phi\longrightarrow\theta),1\rangle} \qquad \frac{}{|\langle((\theta\longrightarrow\phi)\longrightarrow\phi)\longrightarrow((\phi\longrightarrow\theta)\longrightarrow\theta),1\rangle}$$

$$\frac{Y|\langle\theta,a\rangle}{Y|\langle\underline{b}\longrightarrow\theta,b\longrightarrow a\rangle}\ lift(b) \qquad \frac{Y_1|\langle\theta\longrightarrow\phi,a\rangle;Y_2|\langle\theta,b\rangle}{Y_1,Y_2|\langle\phi,a\otimes b\rangle}\ \longrightarrow -out$$

$$\frac{Y_1|\langle\theta,a\rangle;Y_2,\langle\theta,a\rangle|\langle\phi,b\rangle}{Y_1,Y_2|\langle\phi,b\rangle}\ cut$$

where $a,b\in[0,1]$, $\theta,\phi,\psi\in S$ and $Y,Y_1,Y_2\in[0,1]^{S\mathsf{L}}$ are all finite.

Notice that the rules above are all monotone, and so are P-rules. In addition notice that these P-rules are also all *upper semi-continuous* (see [4]), and so $\vdash_{Ł}$ is a P-axiomatization in the sense of this paper **and** PAVELKA's work.

Proposition 9 *For any $X \in [0,1]^{S\text{Ł}}$, $\theta \in S$ and $a \in [0,1]$:*

$$X \vdash_{Ł} \langle \theta, a \rangle \text{ iff } X \models \langle \theta, a \rangle$$

and $\vdash_{Ł}$ is P-sound and P-complete.

Proof. See [7] and [8]. ■

That this P-axiomatization is P-sound and P-complete should be no real surprise, since it essentially consists of the usual axiomatization for the Łukasiewicz $\aleph_0$-valued logic, along with diagram axioms and some structural rules. This ensures that all reasoning essentially occurs within an extension of the Łukasiewicz $\aleph_0$-valued logic, the only difficulty being in ensuring reasoning may be *lifted* to this system and *dropped* from it. The proof detailed in [7] and [8] uses this fact along with some of the underlying model theoretic properties presented in [1] and [2] to achieve its result.

References

[1] CHANG, C. C. *Algebraic Analysis of Many-Valued Logics* Transactions of the American Mathematical Society, Vol. 88, 1958

[2] CHANG, C. C. *A New Proof of the Completeness of the Łukasiewicz Axioms* Transactions of the American Mathematical Society, Vol. 93, 1959

[3] PARIS, J. *Lecture Notes on Inexact Reasoning* Mathematics Department, Manchester University, 1989

[4] PAVELKA, J. *On Fuzzy Logic I: Many Valued Rules of Inference* Zeitshrift für Math. Logik und Grundlagen d. Math., Vol. 25, 1979

[5] PAVELKA, J. *On Fuzzy Logic II: Enriched Residuated Lattices and Properties of Propositional Calculus* Zeitshrift für Math. Logik und Grundlagen d. Math., Vol. 25, 1979

[6] PAVELKA, J. *On Fuzzy Logic III: Semantical Completeness of some Many Valued Propositional Calculi* Zeitshrift für Math. Logik und Grundlagen d. Math., Vol. 25, 1979

[7] PULLEY, C.J. *Some Completeness and Incompleteness Results for Pavelkan-Type Logics on the Unit Interval* MSc Dissertation, Department of Mathematics, University of Manchester, 1989

[8] PULLEY, C.J. *Some Results Concerning Probabilistic Logics and Inference Process Logics* Phd Dissertation, Department of Mathematics, University of Manchester, 1991